SS 2016

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Differential Geometry

Homework 3

Mandatory Exercise 1. (10 points)
Consider the unit sphere S^1 in the complex plane $\mathbb{C}$ and the real projective line $\mathbb{R}P^1$ as the quotient space of S^1 under the identification $z \sim -z$.

(a) Show that $\mathbb{R}P^1$ is a topological manifold and admits a differentiable structure. Describe this differentiable structure explicit by giving an atlas.

(b) Is the map

$$\begin{aligned} \mathbb{R}P^1 &\longrightarrow S^1 \\ [z] &\longmapsto z^2 \end{aligned}$$

a diffeomorphism?

Mandatory Exercise 2. (10 points)

(a) Is there an embedding $S^n \to \mathbb{R}^n$?

(b) Is there an embedding $S^n \times \mathbb{R} \to \mathbb{R}^{n+1}$?

Suggested Exercise 1. (0 points)
Is every bijective differentiable map a diffeomorphism? (Prove this or give a counterexample.)

Suggested Exercise 2. (0 points)
Consider the surface C of a unit cube

$$C := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \colon \max_i(|x_i|) = 1\right\}.$$

(a) Show that C is not a smooth submanifold of $\mathbb{R}^3$.

(b) Show that C is a topological manifold and define a differentiable structure on it.

Suggested Exercise 3. (0 points)
Consider the subspace M of real (4×4)-matrices A fulfilling the equation $A^t DA = D$, where D denotes the diagonal matrix with diagonal entries $(1, 1, 1, -1)$. Show that M is a 6-dimensional submanifold of the space $\mathcal{M}(4 \times 4) \equiv \mathbb{R}^{16}$ of all (4×4)-matrices.

Suggested Exercise 4. (0 points)

(a) Prove that the orthogonal group $O(n)$ is a submanifold of $\mathcal{M}(n \times n) \equiv \mathbb{R}^{n^2}$. What is its dimension?

(b) Describe $T_{\mathrm{Id}} O(n) \subset T_{\mathrm{Id}} \mathbb{R}^{n^2}$.

Suggested Exercise 5. (0 points)
Consider the Möbiusstrip M as a subset of $\mathbb{R}^3$. The boundary of M is homeomorphic to S^1 and can be deformed inside $\mathbb{R}^3$ into the standard unit circle. Find an embedding of the Möbiusstrip M into $\mathbb{R}^3$ with boundary a standard unit circle. Build a paper model.

Suggested Exercise 6. (0 points)
If $f \colon M \to N$ is a diffeomorphism and X, Y are vector fields on M then

$$[df(X), dF(Y)] = df([X, Y]).$$

Hand in: Monday 2nd May
in the exercise session
in Seminar room 2, MI